Int. J. Solids Structures Vol. 28, No. 3, pp. 363–372, 1991
Printed in Great Britain.

0020-7683/91 \$3.00 + .00

THE EIGENSTRAIN FORMULATION FOR CLASSICAL PLATES

SHAOWEN QIN, HUI FAN and TOSHIO MURA
Department of Civil Engineering and Materials Research Center, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208, U.S.A.

(*Received* 13 *August* 1990; *in revised form* 26 *October* 1990)

Abstract—In the present paper, the eigenstrain formulation is developed for classical plates. By employing this approach, Green's function of an isotropic plate is obtained in closed form. The solution for a plate with semi-infinite hinge is derived, which shows a $1/r$ singularity near the tip of the hinge. The Eshelby-type problem, an elliptic inclusion with uniform eigenstrain, is studied and the result has the same features as that of the Eshelby solution for ellipsoidal inclusion in three-dimensional elasticity, i.e., the general stress $M_{\alpha\beta}$ and strain $w_{,\alpha\beta}$ are uniform inside the inclusion. By using the equivalent inclusion method outlined in this paper, an elliptic inhomogeneity which has different material properties from the matrix can be made equivalent to an elliptic inclusion with appropriate uniform eigenstrain, and can therefore be solved readily.

1. INTRODUCTION

Eigenstrain is a generic name of all kinds of inelastic strains, which was originally called transformation strain by Eshelby. Eigenstrain formulation has been widely used in three-dimensional and two-dimensional plane strain problems (see Mura, 1982). Many interesting results were obtained via this approach, which greatly benefitted the studies of micromechanics. However, there is no parallel formulation in thin plate theory, which is one of the most important configurations in applied mechanics. It has been noticed that thin plates with some microstructures become an attractive option in engineering applications. Two well known examples are perforated plates in heat transfer and honeycomb plates in aerospace structures.

In the present paper, the eigenstrain formulation for classical thin plates is developed. For isotropic materials, some important results are obtained in closed form. Green's function obtained in this paper coincides with the existing result, showing $r^2 \ln r$ singularity. As an application of the current approach, a semi-infinite plastic hinge solution is derived, which shows the $1/r$ singularity near the tip of the hinge. This result reminds us of the dislocation solution in two-dimensional plane strain elasticity, which also has $1/r$ singular behavior near the center of the dislocation.

The last problem considered in the present paper is the elliptic inclusion with uniform eigenstrain, which we call the Eshelby-type problem (1957). A uniform distribution of general stress $M_{\alpha\beta}$ and general strain $w_{,\alpha\beta}$ inside the elliptic inclusion is obtained, which is the same as the Eshelby solution for the ellipsoidal inclusion in three-dimensional elasticity. The simplicity of this remarkable result allows us to employ the equivalent inclusion method, so that an elliptic inhomogeneity which has different material properties from the matrix can be made equivalent to an elliptic inclusion with appropriate eigenstrain, and can therefore be solved readily. It is expected that the current approach will be an easier way to solve some problems, such as the effective elastic modulus and failure mechanism of plates with microstructures.

2. EIGENSTRAIN FORMULATION IN PLATE THEORY

2.1. *Classical plate theory*

With the coordinate system shown in Fig. 1, the mathematical formulation for the three-dimensional elasticity is given as:

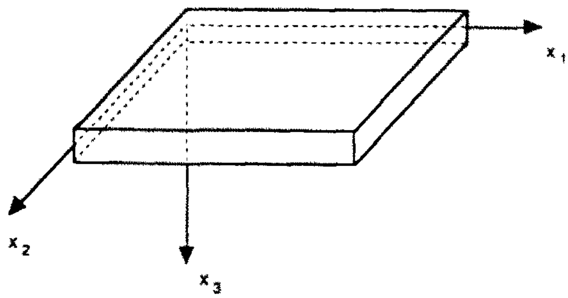

Fig. 1. Configuration of a thin plate.

$$\sigma_{ij,j} = 0 \tag{1}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2}$$

$$\varepsilon_{kl} = \tfrac{1}{2}(u_{k,l}+u_{l,k}) \tag{3}$$

or

$$C_{ijkl}u_{k,lj} = 0 \tag{4}$$

where i, j, k and $l = 1$, 2 and 3, and the Greek letters used in the following are restricted to 1 and 2 only.

Elastic thin plate theory is a two-dimensional approximation of the exact theory of three-dimensional elasticity. The approximation was made based on the so-called Kirchhoff assumption:

$$u_\alpha = -x_3 w_{,\alpha}, \quad u_3 = w(x_1, x_2). \tag{5}$$

Substituting this displacement into Hooke's law, eqn (2), in-plane stresses are expressed in terms of linear functions of x_3:

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\mu}(-x_3 w_{,\gamma\mu}). \tag{6}$$

An integration along the thickness direction provides stress resultants:

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} x_3\sigma_{\alpha\beta}\,\mathrm{d}x_3 = -\frac{h^3}{12}C_{\alpha\beta\gamma\mu}w_{,\gamma\mu} \tag{7a}$$

and

$$Q_\alpha = \int_{-h/2}^{h/2} \sigma_{3\alpha}\,\mathrm{d}x_3. \tag{7b}$$

Equilibrium equations are then rewritten in terms of these resultants as:

$$M_{\alpha\beta,\beta} = Q_\alpha \tag{8a}$$

$$Q_{\alpha,\alpha} = q(x_1, x_2) \tag{8b}$$

where q is the lateral force acting in the x_3 direction. When the upper and lower surfaces are traction free, one will have a homogeneous equation:

$$C_{\alpha\beta\gamma\mu} w_{,\alpha\beta\gamma\mu} = 0. \tag{9}$$

The homogeneous solution of this equation may be assumed in the form of:

$$w(x_1, x_2) = F(z), \quad z = x_1 + p x_2 \tag{10}$$

and

$$w_{,\alpha} = (\delta_{1\alpha} + p\delta_{2\alpha})F'(z)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and $F'(z) = \mathrm{d}F/\mathrm{d}z$. Substituting these formulae into (9) leads to

$$p^4 C_{2222} + 4p^3 C_{2221} + 2p^2(C_{1122} + C_{1212}) + 4pC_{1122} + C_{1111} = 0. \tag{11}$$

It has been proved that:

$$p_3 = \bar{p}_1, \quad p_4 = \bar{p}_2 \tag{12}$$

where the bar stands for complex conjugate.

Then a general solution can be expressed as:

$$w(x_1, x_2) = \mathrm{Re}\,(w^1(z_1) + w^2(z_2)), \tag{13a}$$

where

$$z_1 = x_1 + p_1 x_2 \quad \text{and} \quad z_2 = x_1 + p_2 x_2.$$

When the roots are double root, i.e. $p_1 = p_2$, the solution is:

$$w(x_1, x_2) = \mathrm{Re}\,(w^1(z_1) + \bar{z}_1 w^2(z_1)). \tag{13b}$$

Equation (13) was obtained by Lekhnitskii (1961) for an elastic thin plate.

2.2. *Eigenstrain formulation*

With the formulation given in the previous section, we write the total strain as:

$$\varepsilon_{\alpha\beta} = e_{\alpha\beta} + \varepsilon^*_{\alpha\beta} \tag{14}$$

where $\varepsilon_{\alpha\beta}$ is elastic strain which follows Hooke's law, and $\varepsilon^*_{\alpha\beta}$ is the so-called eigenstrain (see Mura, 1982). From (5) and (6), one has:

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\mu}(-x_3 w_{,\gamma\mu} - \varepsilon^*_{\gamma\mu}). \tag{15}$$

Then the equilibrium equation without lateral loading in the x_3 direction is:

$$C_{\alpha\beta\gamma\mu} w_{,\alpha\beta\gamma\mu} = -C_{\alpha\beta\gamma\mu} k^*_{\gamma\mu,\alpha\beta} \tag{16}$$

where

$$k^*_{\alpha\beta} = \frac{12}{h^3}\int_{-h/2}^{h/2} x_3 \varepsilon^*_{\alpha\beta}\,\mathrm{d}x_3.$$

Notice that only the antisymmetric part of the eigenstrain with respect to the middle plane ($x_3 = 0$) contributes to the plate bending problem. The symmetric part corresponds to the in-plane deformation which is called the plane stress problem.

Assuming that $k^*_{\gamma\mu}$ is given by the Fourier integral form,

$$k^*_{\gamma\mu}(x_1, x_2) = \int\int_{-\infty}^{\infty} \bar{k}^*_{\gamma\mu}(\xi_1, \xi_2)\exp(i\xi_\alpha x_\alpha)\,d\xi_1\,d\xi_2 \tag{17a}$$

and

$$w(x_1, x_2) = \int\int_{-\infty}^{\infty} \bar{w}(\xi_1, \xi_2)\exp(i\xi_\alpha x_\alpha)\,d\xi_1\,d\xi_2. \tag{17b}$$

By using (16), we can obtain:

$$\bar{w}(\xi_1, \xi_2) = \frac{C_{\alpha\beta\gamma\mu}\bar{k}^*_{\gamma\mu}\xi_\alpha\xi_\beta}{C_{\alpha\beta\gamma\mu}\xi_\alpha\xi_\beta\xi_\gamma\xi_\mu}. \tag{18}$$

Furthermore:

$$w(x_1, x_2) = \int\int_{-\infty}^{\infty} \frac{C_{\alpha\beta\gamma\mu}\bar{k}^*_{\gamma\mu}\xi_\alpha\xi_\beta}{C_{\alpha\beta\gamma\mu}\xi_\alpha\xi_\beta\xi_\gamma\xi_\mu}\exp(i\xi_\alpha x_\alpha)\,d\xi_1\,d\xi_2 \tag{19}$$

where

$$\bar{k}^*_{\gamma\mu}(\xi_1, \xi_2) = \frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} k^*_{\gamma\mu}(x_1, x_2)\exp(-i\xi_\alpha x_\alpha)\,dx_1\,dx_2. \tag{20}$$

A general solution is then obtained as:

$$w(x_1, x_2) = \frac{1}{4\pi^2}\int\int\int\int_{-\infty}^{\infty} \frac{C_{\alpha\beta\gamma\mu}k^*_{\gamma\mu}(x'_1, x'_2)\xi_\alpha\xi_\beta}{C_{\alpha\beta\gamma\mu}\xi_\alpha\xi_\beta\xi_\gamma\xi_\mu}\exp(i\xi_\alpha(x_\alpha - x'_\alpha))\,d\xi_1\,d\xi_2\,dx'_1\,dx'_2. \tag{21}$$

Green's function may be defined from the above equation as:

$$G(x_1 - x'_1, x_2 - x'_2) = \frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} \frac{\exp(i\xi_\alpha(x_\alpha - x'_\alpha))}{C_{\alpha\beta\gamma\mu}\xi_\alpha\xi_\beta\xi_\gamma\xi_\mu}\,d\xi_1\,d\xi_2. \tag{22}$$

Therefore,

$$w(x_1, x_2) = -\int\int_{-\infty}^{\infty} C_{\alpha\beta\gamma\mu}k^*_{\gamma\mu}(x'_1, x'_2)G_{,\alpha\beta}(x_1 - x'_1, x_2 - x'_2)\,dx'_1\,dx'_2. \tag{23}$$

For isotropic materials,

$$C_{\alpha\beta\gamma\mu}\xi_\alpha\xi_\beta\xi_\gamma\xi_\mu = (\lambda + 2\mu)(\xi_1^2 + \xi_2^2)^2$$

where λ and μ are the Lamé constants.

With the integrals given in the Appendix, Green's function for isotropic plates is obtained in closed form as:

$$G(x_1 - x'_1, x_2 - x'_2) = \frac{1}{16\pi(\lambda + 2\mu)}((x_1 - x'_1)^2 + (x_2 - x'_2)^2) \times (\log((x_1 - x'_1)^2 + (x_2 - x'_2)^2) - C_0) \tag{24}$$

where C_0 is an arbitrary constant.

With the following expression for isotropic materials,

$$C_{\alpha\beta;\gamma\mu}\xi_\alpha\xi_\beta k^*_{\gamma\mu} = \lambda\xi_\alpha\xi_\alpha k^*_{\beta\beta}+2\mu(\xi_1^2 k^*_{11}+\xi_2^2 k^*_{22}+(k^*_{12}+k^*_{21})\xi_1\xi_2)$$

and again the integrals given in the Appendix, eqn (21) becomes:

$$w(x_1,x_2) = \frac{1}{4\pi(\lambda+2\mu)}\int_\Omega \Bigg[(\lambda+\mu)k^*_{\alpha\alpha}\log((x_1-x_1')^2+(x_2-x_2')^2) + \frac{2\mu}{(x_1-x_1')^2+(x_2-x_2')^2}(k^*_{11}(x_1-x_1')^2 + k^*_{22}(x_2-x_2')^2+(k^*_{12}+k^*_{21})(x_1-x_1')(x_2-x_2'))\Bigg]\mathrm{d}x_1'\,\mathrm{d}x_2' \quad (25)$$

where Ω is the domain with non-zero eigenstrain. Equation (25) can also be reached by using (23) and (24) directly.

3. SOME APPLICATIONS OF THE EIGENSTRAIN FORMULATION

In this section, two important examples are discussed based on the eigenstrain formulation.

3.1. *A semi-infinite plastic hinge*

Consider a plate with a semi-infinite plastic hinge as shown in Fig. 2. The jump conditions for the plastic hinge are:

$$[w] = 0, \quad [w_{,2}] = \theta H(-x_1), \quad \text{at} \quad x_2 = 0 \quad (26)$$

where θ is the jump of rotation across the hinge, and $H(x)$ is the Heaviside function.

The eigenstrain can be written as:

$$k^*_{21} = \frac{\theta}{2}H(-x_1)\delta(x_2). \quad (27)$$

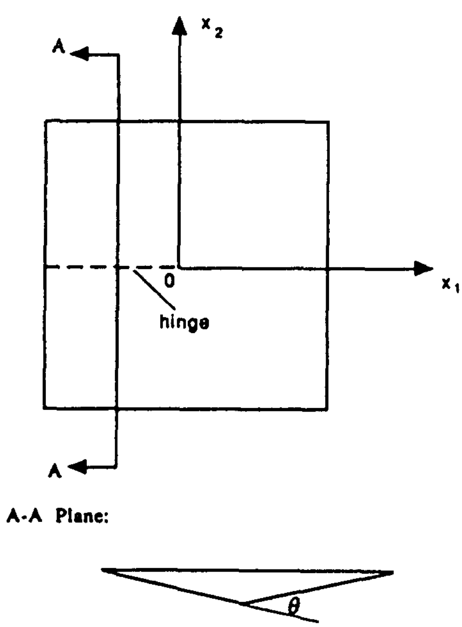

Fig. 2. A semi-infinite plastic hinge.

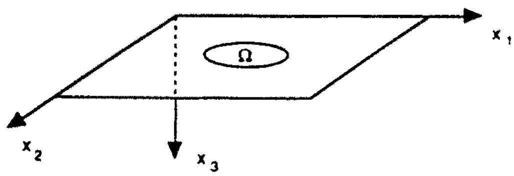

Fig. 3. An elliptic inclusion with uniform eigenstrain.

Here $\delta(x_2)$ is the Dirac delta function. With Green's function (24), one can obtain:

$$w(x_1,x_2) = \frac{\theta\mu}{2\pi(\lambda+2\mu)}\int\int_{-\infty}^{\infty} H(-x'_1)\delta(x'_2)\frac{(x_1-x'_1)(x_2-x'_2)}{(x_1-x'_1)^2+(x_2-x'_2)^2}\,dx'_1\,dx'_2 \tag{28}$$

and the following results can then be reached:

$$w_{,11} = \frac{\theta\mu}{2\pi(\lambda+2\mu)}\frac{x_2(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2} \tag{29a}$$

$$w_{,22} = \frac{\theta\mu}{2\pi(\lambda+2\mu)}\frac{x_2(3x_1^2+x_2^2)}{(x_1^2+x_2^2)^2} \tag{29b}$$

$$w_{,12} = \frac{\theta\mu}{2\pi(\lambda+2\mu)}\frac{x_1(x_1^2-x_2^2)}{(x_1^2+x_2^2)^2}. \tag{29c}$$

The corresponding stress field can be readily obtained by putting the above strain field into eqn (6). An interesting feature of the solution is that the general strain $w_{,\alpha\beta}$ and stress $M_{\alpha\beta}$ expressions have $1/r$ singularity, which is just like the behavior of a straight dislocation in the plane strain case. The results given above could be useful for the plastic limit analysis of thin plates.

3.2. *Elliptic inclusion with uniform eigenstrain*

Eshelby's solution (1957) of an ellipsoidal inclusion in an infinite elastic medium is thought of as a corner stone of micromechanics. The formulation in this section is an extension of the Eshelby solution to thin plates under bending. Figure 3 shows an elliptic inclusion with uniform eigenstrain sitting in an infinite plate. The general stress and strain in the plate theory are $M_{\alpha\beta}$ and $w_{,\alpha\beta}$, respectively. We expect them to be uniform inside the inclusion, as in the Eshelby solution.

The domain of the inclusion, Ω, is defined as:

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2} = 1.$$

We start from eqn (25). By taking derivatives of w with respect to x_μ, and using a polar coordinate system with origin at (x_1, x_2) (see Fig. 4), the following expression is reached:

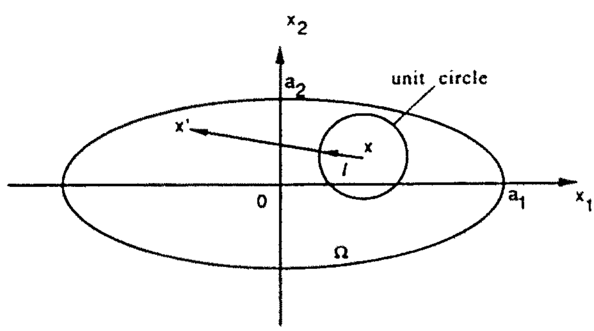

Fig. 4. Illustration of the integration technique.

$$w_{,p} = \pi A \int\int_{\Omega} \Bigg((\lambda+\mu)k^*_{\beta\beta}\frac{2l_1\delta_{1p}+2l_2\delta_{2p}}{r} + \frac{2\mu}{r}(k^*_{11}(2l_1l_2^2\delta_{1p}-2l_1^2l_2\delta_{2p})+k^*_{22}(2l_1^2l_2\delta_{2p}-2l_1l_2^2\delta_{1p}) + (k^*_{12}+k^*_{21})((l_1^3-l_1l_2^2)\delta_{2p}+(l_2^3-l_2l_1^2)\delta_{1p})) \Bigg) \mathrm{d}x'_1\,\mathrm{d}x'_2 \quad (30)$$

where

$$A = \frac{1}{4\pi(\lambda+2\mu)}, \quad r^2 = (x_1-x'_1)^2+(x_2-x'_2)^2, \quad rl_1 = x'_1-x_1, \quad rl_2 = x'_2-x_2$$

and l is the unit vector along the $\mathbf{x}-\mathbf{x}'$ direction. With $\mathrm{d}x'_1\,\mathrm{d}x'_2 = r\,\mathrm{d}r\,\mathrm{d}\theta$, the integrand of (30) becomes independent of r, i.e.

$$w_{,p} = 2\pi A\int_0^{r(l_1,l_2)}\int_0^{2\pi} F_p(l_1,l_2)\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi A\int_0^{2\pi} F_p(l_1,l_2)r(l_1,l_2)\,\mathrm{d}\theta \quad (31)$$

where

$$F_p(l_1,l_2) = (\lambda+\mu)k^*_{\alpha\alpha}(l_1\delta_{1p}+l_2\delta_{2p}) + 2\mu(k^*_{11}(l_1l_2^2\delta_{1p}-l_1^2l_2\delta_{2p})+k^*_{22}(l_1^2l_2\delta_{2p}-l_1l_2^2\delta_{1p}) + (k^*_{12}+k^*_{21})((l_1^3-l_1l_2^2)\delta_{2p}+(l_2^3-l_2l_1^2)\delta_{1p})/2). \quad (32)$$

When point x is located inside the inclusion Ω, the integral in eqn (31) is explicitly performed. From the elliptic equation which represents the boundary of Ω, $r(l_1,l_2)$ can be found as follows:

$$r(l_1,l_2) = -f/g \pm \sqrt{f^2/g^2+e/g}$$

where

$$f = \frac{l_1x_1}{a_1^2}+\frac{l_2x_2}{a_2^2}$$

$$g = \frac{l_1^2}{a_1^2}+\frac{l_2^2}{a_2^2}$$

$$e = 1-\frac{x_1^2}{a_1^2}-\frac{x_2^2}{a_2^2}.$$

When $r(l_1,l_2)$ is inserted into eqn (31), the term $\sqrt{f^2/g^2+e/g}$ can be omitted, since it is an even function of l, while $F_p(l_1,l_2)$ is odd. Thus,

$$w_{,p} = -2\pi A\int_0^{2\pi} F_p(l_1,l_2)\frac{f(x_1,x_2;l_1,l_2)}{g(l_1,l_2)}\,\mathrm{d}\theta. \quad (33)$$

Rewriting f as:

$$f = \lambda_m x_m,$$

eqn (33) becomes:

$$w_{,p} = -2\pi A x_m \int_0^{2\pi} F_p(l_1, l_2) \frac{\lambda_m}{g(l_1, l_2)} \, d\theta \tag{34}$$

where

$$\lambda_1 = l_1/a_1^2, \quad \lambda_2 = l_2/a_2^2.$$

Now we can take another derivative of $w_{,p}$ with respect to x_q to get the general strain:

$$w_{,pq} = -2\pi A \int_0^{2\pi} \frac{\lambda_q F_p}{g} \, d\theta. \tag{35}$$

Equation (35) is the desired results which shows that the general strains $w_{,pq}$ inside the inclusion are constant. Consequently, the moment components, $M_{\alpha\beta}$, defined in (7a) are also constants in Ω.

The integrals in the above equation are carried out as follows:

$$w_{,11} = -\frac{1-2\nu}{2\pi\mu(1-\nu)} \int_{-\pi/2}^{\pi/2} \frac{l_1}{a_1^2(l_1^2/a_1^2 + l_2^2/a_2^2)} \left(\frac{\mu}{1-2\nu}(k_{11}^* + k_{22}^*) l_1 + 2\mu l_1 l_2^2((k_{11}^* - k_{22}^*) + k_{12}^*(l_2^3 - l_1^2 l_2)) \right) d\theta \tag{36a}$$

$$w_{,22} = -\frac{1-2\nu}{2\pi\mu(1-\nu)} \int_{-\pi/2}^{\pi/2} \frac{l_2}{a_2^2(l_1^2/a_1^2 + l_2^2/a_2^2)} \left(\frac{\mu}{1-2\nu}(k_{11}^* + k_{22}^*) l_2 + 2\mu l_1^2 l_2((k_{22}^* - k_{11}^*) + k_{12}^*(l_1^3 - l_2^2 l_1)) \right) d\theta \tag{36b}$$

$$w_{,12} = -\frac{1-2\nu}{2\pi\mu(1-\nu)} \int_{-\pi/2}^{\pi/2} \frac{l_2^4 - l_1^2 l_2^2}{a_2^2(l_1^2/a_1^2 + l_2^2/a_2^2)} 2\mu k_{12}^* \, d\theta \tag{36c}$$

$$w_{,21} = -\frac{1-2\nu}{2\pi\mu(1-\nu)} \int_{-\pi/2}^{\pi/2} \frac{l_1^4 - l_1^2 l_2^2}{a_1^2(l_1^2/a_1^2 + l_2^2/a_2^2)} 2\mu k_{21}^* \, d\theta \tag{36d}$$

where ν is Poisson's ratio.

For the convenience of further application discussed in the next section, we now relate $w_{,\alpha\beta}$ with eigenstrain by a tensor S,

$$w_{,\alpha\beta} = S_{\alpha\beta\gamma\mu} k_{\gamma\mu}^* \tag{37}$$

where

$$S_{1111} = -\frac{1}{2\pi(1-\nu)} \frac{I_1}{a_1^2} - \frac{1-2\nu}{\pi(1-\nu)} \frac{I_{12}}{a_1^2}$$

$$S_{2222} = -\frac{1}{2\pi(1-\nu)} \frac{I_2}{a_2^2} - \frac{1-2\nu}{\pi(1-\nu)} \frac{I_{12}}{a_2^2}$$

$$S_{1122} = -\frac{1}{2\pi(1-\nu)} \frac{I_1}{a_1^2} + \frac{1-2\nu}{\pi(1-\nu)} \frac{I_{12}}{a_1^2}$$

$$S_{2211} = -\frac{1}{2\pi(1-\nu)} \frac{I_2}{a_2^2} + \frac{1-2\nu}{\pi(1-\nu)} \frac{I_{12}}{a_2^2}$$

$$S_{1212} = -\frac{1-2\nu}{2\pi(1-\nu)} \frac{I_{22} - I_{12}}{a_2^2}$$

$$S_{2121} = -\frac{1-2\nu}{2\pi(1-\nu)}\frac{I_{11}-I_{12}}{a_1^2}.$$

The rest of the components are zero. The I values used in the above expressions are:

$$I_1 = \int_{-\pi/2}^{\pi/2} \frac{l_1^2}{g}\,d\theta = \frac{\pi a_1^2 a_2^2}{a_2(a_1+a_2)}$$

$$I_2 = \int_{-\pi/2}^{\pi/2} \frac{l_2^2}{g}\,d\theta = \frac{\pi a_1^2 a_2^2}{a_1(a_1+a_2)}$$

$$I_{11} = \int_{-\pi/2}^{\pi/2} \frac{l_1^4}{g}\,d\theta = \frac{\pi a_1^2 a_2(2a_1^3+a_2^3-3a_1^2 a_2)}{2(a_2^2-a_1^2)^2}$$

$$I_{22} = \int_{-\pi/2}^{\pi/2} \frac{l_2^4}{g}\,d\theta = \frac{\pi a_1 a_2^2(2a_2^3+a_1^3-3a_2^2 a_1)}{2(a_1^2-a_2^2)^2}$$

$$I_{12} = \int_{-\pi/2}^{\pi/2} \frac{l_1^2 l_2^2}{g}\,d\theta = \frac{\pi a_1^2 a_2^2}{2(a_1+a_2)^2}.$$

In the case of a circular inclusion, i.e. $a_1 = a_2 = a$, the integrals are reduced to:

$$I_1 = I_2 = \frac{\pi a^2}{2}$$

$$I_{11} = I_{22} = \frac{3\pi a^2}{8}$$

$$I_{12} = \frac{\pi a^2}{8}.$$

The anisotropic inclusion solution is expected to have the same features as that of the isotropic solution discussed above. Further work will be devoted to this subject.

4. EQUIVALENT INCLUSION METHOD

The solution obtained in the previous section is very useful for inhomogeneities which have different elastic moduli from the matrix. We can find an inclusion of the same shape with appropriate eigenstrains which will cause exactly the same stress and strain fields as that of the inhomogeneity. This procedure is called the equivalent inclusion method, which has been extensively developed by Mura (1982) in three-dimensional problems based upon the Eshelby solution (1957).

Consider an infinitely extended plate with elastic moduli $C_{\alpha\beta\gamma\mu}$ containing an elliptic domain with the elastic moduli $C^*_{\alpha\beta\gamma\mu}$ (see Fig. 3). Let us denote the far field moment as $M^0_{\alpha\beta}$ and the corresponding strain $w^0_{,\alpha\beta}$. The disturbance caused by the inhomogeneity is $M_{\alpha\beta}$ and $w_{,\alpha\beta}$ respectively. Considering a point inside the inhomogeneity, we have two expressions for moments in terms of different moduli:

$$M^0_{\alpha\beta}+M_{\alpha\beta} = -\frac{h^3}{12}C_{\alpha\beta\gamma\mu}(w^0_{,\gamma\mu}+w_{,\gamma\mu}-k^*_{\gamma\mu}) \tag{38}$$

or

$$M^0_{\alpha\beta}+M_{\alpha\beta} = -\frac{h^3}{12}C^*_{\alpha\beta\gamma\mu}(w^0_{,\gamma\mu}+w_{,\gamma\mu}). \tag{39}$$

For elliptic inclusion, recall that

$$w_{,\alpha\beta} = S_{\alpha\beta\gamma\mu} k^{*}_{\gamma\mu}. \tag{37}$$

We then have the following expression:

$$C_{\alpha\beta\gamma\mu}(w^{0}_{\gamma\mu} + S_{\gamma\mu pq} k^{*}_{pq} - k^{*}_{\gamma\mu}) = C^{*}_{\alpha\beta\gamma\mu}(w^{0}_{\gamma\mu} + S_{\gamma\mu pq} k^{*}_{pq}). \tag{40}$$

This equation determines the eigenstrains of the equivalent inclusion.

5. DISCUSSION AND FURTHER WORK

The formulation in the previous sections extends Eshelby's solution to the thin plate theory. It is known that the plate theory is an approximated formulation based on three-dimensional elasticity. The so-called effective shear force concept is needed to make this Boundary Value Problem a well defined one. According to the assumption of the thin plate, the stress components $\sigma_{3\alpha}$ are higher order quantities than in-plane components $\sigma_{\alpha\beta}$. In the previous derivation only $M_{\alpha\beta}$ showed the property of Eshelby-type solution, while Q_{α} was not involved in the calculation. This will not be the case when energies are considered where $M_{\alpha\beta} w_{,\alpha\beta}$ and $Q_{\alpha} w_{,\alpha}$ are of the same order.

Further work can be done along these lines. For example, a crack in a thin plate under bending can be examined by using the present method. It is the authors' opinion that the eigenstrain approach could be an easier way to study certain mechanical behavior of thin plates with microstructures.

Acknowledgement—This work was supported by the U.S. Army Research Office under grant no. DAAL03-89-K0019.

REFERENCES

Campbell, G. A. and Foster, R. M. (1948). *Fourier Integrals for Practical Applications*. D. Van Nostrand, New York.

Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. R. Soc.* **A241**, 376.

Lekhnitskii, S. G. (1961). *Anisotropic Plates*. Gordon & Breach, New York.

Mura, T. (1982). *Micromechanics of Defects in Solids*. Martinus Nijhoff, London.

APPENDIX

The integrals used to derive Green's function are obtained from *Fourier Integrals for Practical Applications* by Campbell and Foster (1948):

$$\int\int_{-\infty}^{\infty} \frac{\exp(i(\xi_1 x_1 + \xi_2 x_2))}{\xi_1^2 + \xi_2^2} \, d\xi_1 \, d\xi_2 = -\pi \log(x_1^2 + x_2^2) \tag{A1}$$

$$\int\int_{-\infty}^{\infty} \frac{\xi_1 \xi_2 \exp(i(\xi_1 x_1 + \xi_2 x_2))}{(\xi_1^2 + \xi_2^2)^2} \, d\xi_1 \, d\xi_2 = -\frac{\pi x_1 x_2}{x_1^2 + x_2^2} \tag{A2}$$

$$\int\int_{-\infty}^{\infty} \frac{\xi_2^2 \exp(i(\xi_1 x_1 + \xi_2 x_2))}{(\xi_1^2 + \xi_2^2)^2} \, d\xi_1 \, d\xi_2 = -\frac{\pi}{2} \log(x_1^2 + x_2^2) - \frac{\pi x_2^2}{x_1^2 + x_2^2}. \tag{A3}$$